1. If the variance \( \sigma_x^2 \) of \( d(n) = x(n) - x(n - 1) \) is one-tenth the variance \( \sigma_x^2 \) of a stationary zero-mean discrete-time signal \( x(n) \), then the normalized autocorrelation function \( R_{xx}(k)/\sigma_x^2 \) at \( k = 1 \) is
   (a) 0.95  
   (b) 0.90  
   (c) 0.10  
   (d) 0.05

   \[ \text{[GATE 2002: 2 Marks]} \]

   \text{Soln. The variance } \sigma_x^2 = E[(X - \mu_X)^2] 
   \text{Where } \mu_X (mean value) = 0

   \[ \sigma_d^2 = E[(X(n) - X(n - 1))^2] \]

   \[ \sigma_d^2 = E[X(n)]^2 + E[X(n - 1)]^2 - 2E[X(n)X(n - 1)] \]

   \[ \frac{\sigma_x^2}{10} = \sigma_x^2 + \sigma_x^2 - 2R_{xx}(1) \]

   \[ \sigma_x^2 = 20\sigma_x^2 - 20R_{xx}(1) \]

   \[ \frac{R_{xx}}{\sigma_x^2} = \frac{19}{20} = 0.95 \]

   Option (a)

2. Let \( Y \) and \( Z \) be the random variables obtained by sampling \( X(t) \) at \( t = 2 \) and \( t = 4 \) respectively. Let \( W = Y - Z \). The variance of \( W \) is
   (a) 13.36  
   (b) 9.36  
   (c) 2.64  
   (d) 8.00

   \[ \text{[GATE 2003: 2 Marks]} \]
Soln. \( W = Y - Z \)  

Given \( R_{XX(\tau)} = 4(e^{-0.2|\tau|} + 1) \)

Variance[\( W \)] = \( E[Y - Z]^2 \)

\[ \sigma^2_W = E[Y^2] + E[Z^2] - 2E[YZ] \]

\( Y \) and \( Z \) are samples of \( X(t) \) at \( t = 2 \) and \( t = 4 \)

\[ E[Y^2] = E[X^2(2)] = R_{XX(0)} \]

\[ = 4(e^{-2|0|} + 1) = 8 \]

\[ E[Z^2] = E[X^2(4)] = 4(e^{-0.2|0|} + 1) = 8 \]

\[ E[YZ] = R_{XX(2)} = 4(e^{-0.2(4-2)} + 1) = 6.68 \]

\[ \sigma^2_W = 8 + 8 - 2 \times 6.68 = 2.64 \]

Option (c)

3. The distribution function \( F_X(x) \) of a random variable \( X \) is shown in the figure. The probability that \( X = 1 \) is

\[ F_X(X) \]

\[ (a) \text{ Zero} \quad (b) \text{ 0.25} \quad (c) \text{ 0.55} \quad (d) \text{ 0.30} \]

\[ \text{[GATE 2004: 1 Mark]} \]

Soln. The probability that \( X = 1 = F_X(x = 1^+) - F_X(x = 1^-) \)

\[ P(x = 1) = 0.55 - 0.25 = 0.30 \]

Option (d)
4. If $E$ denotes expectation, the variance of a random variable $X$ is given by

(a) $E[X^2] - E^2[X]$  
(b) $E[X^2] + E^2[X]$  
(c) $E[X^2]$  
(d) $E^2[X]$  

\[ \text{[GATE 2007: 1 Mark]} \]

Soln. The variance of random variable $X$

$\sigma_X^2 = E[(X - \mu_X)^2]$ 

Where $\mu_X$ is the mean value $= E[X]$

$\sigma_X^2 = E[X^2] + E[\mu_X]^2 - 2 \mu_X E[X]$ 

$\quad = E[X^2] + \mu_X^2 - 2 \mu_X \mu_X$ 

$\quad = E[X^2] - \mu_X^2$ 

$\quad = \text{mean square value} - \text{square of mean value}$

Option (a)

5. If $R(\tau)$ is the auto-correlation function of a real, wide-sense stationary random process, then which of the following is NOT true?

(a) $R(\tau) = R(-\tau)$  
(b) $|R(\tau)| \leq R(0)$  
(c) $R(\tau) = -R(-\tau)$  
(d) The mean square value of the process is $R(0)$  

\[ \text{[GATE 2007: 1 Mark]} \]

Soln. If all the statistical properties of a random process are independent of time, it is known as stationary process.

The autocorrelation function is the measure of similarity of a function with its delayed replica.

\[ R(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t - \tau) f^*(t) \, dt \]
for \( \tau = 0 \), \( R(0) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t) f^*(t) dt \)

\[
= \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt
\]

\( R(0) \) is the average power \( P \) of the signal.

\( R(\tau) = R^*(-\tau) \) exhibits conjugate symmetry

\( R(\tau) = R(-\tau) \) for real function

\( R(0) \geq R(\tau) \) for all \( \tau \)

\( R(\tau) = -R(-\tau) \) is not true (since it has even symmetry)

Option (c)

6. If \( S(f) \) is the power spectral density of a real, wide-sense stationary random process, then which of the following is ALWAYS true?

(a) \( S(0) \geq S(f) \)

(b) \( S(f) \geq 0 \)

(c) \( S(-f) = -S(f) \)

(d) \( \int_{-\infty}^{\infty} S(f) df = 0 \)

[\text{GATE 2007: 1 Mark}]

\text{Soln.} \) Power spectral density is always positive

\( S(f) \geq 0 \)

Option (b)

7. \( P_X(x) = M \exp(-2|x|) + N \exp(-3|x|) \) is the probability density function for the real random variable \( X \) over the entire \( X \) axis \( M \) and \( N \) are both positive real numbers. The equation relating \( M \) and \( N \) is

(a) \( M + \frac{2}{3} N = 1 \)

(b) \( 2M + \frac{1}{3} N = 1 \)

(c) \( M + N = 1 \)

(d) \( M + N = 3 \)

[\text{GATE 2008: 2 Marks}]
Soln.

\[
\int_{-\infty}^{\infty} P_X(x) \, dx = 1
\]

\[
\int_{-\infty}^{\infty} (M \, e^{-2x} + N \, e^{-3x}) \, dx = 1
\]

\[
\int_{0}^{\infty} (M \, e^{-2x} + N \, e^{-3x}) \, dx = \frac{1}{2}
\]

\[
\frac{M \, e^{-2x}}{-2} \bigg|_{0}^{\infty} + \frac{N \, e^{-3x}}{-3} \bigg|_{0}^{\infty} = \frac{1}{2}
\]

\[
\frac{M}{2} + \frac{N}{3} = \frac{1}{2}
\]

or,

\[
M + \frac{2N}{3} = 1
\]

Option (a)

8. A white noise process \( X(t) \) with two-sided power spectral density \( 1 \times 10^{-10} \, W/Hz \) is input to a filter whose magnitude squared response is shown below.

The power of the output process \( y(t) \) is given by

(a) \( 5 \times 10^{-7} W \)
(b) \( 1 \times 10^{-6} W \)
(c) \( 2 \times 10^{-6} W \)
(d) \( 1 \times 10^{-5} W \)

\[\text{[GATE 2009: 1 Mark]}\]
Soln. Power spectral density of white noise at the input of a filter = $G_i(f)$

$$G_i(f) = 1 \times 10^{-10} (W/Hz)$$

PSD at the output of a filter

$$G_0(f) = |H(f)|^2 G_i(f)$$

$$= \frac{1}{2} (2 \times 10 \times 10^3 \times 1) \times 10^{-10}$$

$$= 10^{-6} W$$

Option (b)

9. Consider two independent random variables $X$ and $Y$ with identical distributions. The variables $X$ and $Y$ take value 0, 1 and 2 with probabilities $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{4}$ respectively. What is the conditional probability $(X + Y = 2 | X - Y = 0)$?

(a) 0  
(b) $\frac{1}{16}$  
(c) $\frac{1}{6}$  
(d) 1

[GATE 2009: 2 Marks]

Soln.

$$P(X = 0) = P(Y = 0) = \frac{1}{2}$$

$$P(X = 1) = P(Y = 1) = \frac{1}{4}$$

$$P(X = 2) = P(Y = 2) = \frac{1}{4}$$

$$P(X - Y = 0) = P(X = 0,Y = 0) + P(X = 1,Y = 1)$$

$$+ P(X = 2,Y = 2) = \frac{1}{2} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} = \frac{6}{16}$$

$$P(X + Y = 2) = P(X = 1,Y = 1) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$$
\[ P(X + Y = 2 \mid x - y = 0) = \frac{1}{16} \div \frac{6}{16} = \frac{1}{6} \]

Option (c)

10. \(X(t)\) is a stationary random process with autocorrelation function \(R_X(\tau) = \exp(-\pi \tau^2)\). This process is passed through the system below. The power spectral density of the output process \(Y(t)\) is

\[
\begin{align*}
H(f) &= j2\pi f \\
X(t) &\rightarrow H(f) = j2\pi f \rightarrow + \rightarrow \sum \rightarrow Y(t) \\
\end{align*}
\]

(a) \((4\pi^2 f^2 + 1) \exp(-\pi f^2)\)
(b) \((4\pi^2 f^2 - 1) \exp(-\pi f^2)\)
(c) \((4\pi^2 f^2 + 1) \exp(-\pi f)\)
(d) \((4\pi^2 f^2 - 1) \exp(-\pi f)\)

[\text{GATE 2011: 2 Marks}]

Soln.

\[
Y(f) = j2\pi f \ X(f) - X(f)
\]

PSD \(S_Y(f) = |(j2\pi f - 1)^2|S_X(f)\)

\[
S_X(f) = FT\{R_X(\tau)\} = FT(e^{-\pi \tau^2}) = e^{-\pi f^2}
\]

\[
S_Y(f) = (4\pi^2 f^2 + 1)e^{-\pi f^2}
\]

Option (a)
11. Two independent random variables $X$ and $Y$ are uniformly distributed in the interval $[-1, 1]$. The probability that $\max [X, Y]$ is less than $1/2$ is 

(a) $3/4$  \hspace{1cm} (c) $1/4$ 
(b) $9/16$  \hspace{1cm} (d) $2/3$

\[ \text{[GATE 2012: 1 Mark]} \]

**Soln.**

\[-1 \leq X \leq 1 \text{  and  } -1 \leq Y \leq 1 \]

The region in which maximum of $[X, Y]$ is less than $1/2$ is shown as shaded region inside the rectangle.

\[
P \left[ \max (X, Y) < \frac{1}{2} \right] = \frac{\text{Area of shaded region}}{\text{Area of entire region}}
\]

\[
= \frac{\frac{3}{2} \times \frac{3}{2}}{2 \times 2} = \frac{9}{4 \times 4}
\]

\[
= \frac{9}{16}
\]

Option (b)
12. A power spectral density of a real process $X(t)$ for positive frequencies is shown below. The values of $[E[X^2(t)] \text{ and } |E[X(t)]|]$ respectively are

\[
S_X(\omega) \quad \text{with} \quad 400 \delta(\omega - 10^4)
\]

(a) $6000/\pi, 0$
(b) $6400/\pi, 0$
(c) $6400/\pi, 20/(\pi\sqrt{2})$
(d) $6000/\pi, 20/(\pi\sqrt{2})$

[GATE 2012: 1 Mark]

Soln. The mean square value of a stationary process equals the total area under the graph of power spectral density

\[
E[X^2(t)] = \int_{-\infty}^{\infty} S_X(f) df
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) d\omega
\]

\[
= \frac{2}{2\pi} \int_{0}^{\infty} S_X(\omega) d\omega
\]

\[
= \frac{1}{\pi} \left[ \text{area under the triangle} + \text{integration under delta function} \right]
\]

\[
= \frac{1}{\pi} \left[ 2 \left( \frac{1}{2} \times 1 \times 6 \times 10^3 \right) + 400 \right]
\]
\[
\frac{6400}{\pi}
\]

\[|E[X(t)]| \text{ is the absolute value of mean of signal } X(t) \text{ which is also equal to value of } X(\omega) \text{ at } \omega = 0\]

From PSD
\[S_X(\omega)|_{\omega=0} = 0\]

\[|X(\omega)|^2 = 0\]

\[|X(\omega)| = 0\]

Option (b)

13. Let U and V be two independent zero mean Gaussian random variables of variances \(\frac{1}{4}\) and \(\frac{1}{9}\) respectively. The probability \(P(3V \geq 2U)\) is
(a) 4/9  
(b) 1/2  
(c) 2/3  
(d) 5/9

[GATE 2013: 2 Marks]

Soln.

\[P(3V - 2U) = P(3V - 2U \geq 0)\]

\[= P(W \geq 0)\]

\[W = 3V - 2U\]
W is the Gaussian Variable with zero mean having pdf curve as shown below

\[ P(W \geq 0) = \frac{1}{2} \text{(area under the curve from 0 to } \infty) \]

Option (b)

14. Let \( X_1, X_2, \text{ and } X_3 \) be independent and identically distributed random variables with the uniform distribution on \([0,1]\). The probability \( P\{X_1 \text{ is the largest}\} \) is ________

\[ P\{X_1 \text{ is the largest}\} = \frac{1}{3} P\{X_1\} = \frac{1}{3} P\{X_2\} = \frac{1}{3} P\{X_3\} \]

\[ P(X_1) + P(X_2) + P(X_3) = 1 \]

\[ 3P(X_1) = 1 \]

\[ P(X_1) = \frac{1}{3} \]

15. Let \( X \) be a real-valued random variable with \( E[X] \) and \( E[X^2] \) denoting the mean values of \( X \) and \( X^2 \), respectively. The relation which always holds

(a) \( (E[X])^2 \geq E[X^2] \)  
(b) \( E[X^2] \geq (E[X])^2 \)

(c) \( E[X^2] = (E[X])^2 \)  
(d) \( E[X]^2 \geq (E[X])^2 \)

\[ \text{Variance is always positive so } E[X^2] \geq [E(X)]^2 \]

And can be zero

Option (b)
16. Consider a random process $X(t) = \sqrt{2} \sin(2\pi t + \phi)$, where the random phase $\phi$ is uniformly distributed in the interval $[0,2\pi]$. The autocorrelation $E[ X(t_1) X(t_2) ]$ is

(a) $\cos[2\pi(t_1 + t_2)]$

(b) $\sin[2\pi(t_1 - t_2)]$

(c) $\sin[2\pi(t_1 + t_2)]$

(d) $\cos[2\pi(t_1 - t_2)]$

[**GATE 2014: 2 Marks**]

**Soln.**

$$E[ X(t_1) X(t_2) ] = E[ A \sin(2\pi t_1 + \phi) \times A \sin(2\pi t_2 + \phi) ]$$

$$= \frac{A^2}{2} E[ \cos 2\pi(t_1 - t_2) - \cos 2\pi(t_1 + t_2 + 2\phi) ]$$

$$= \frac{A^2}{2} \cos 2\pi(t_1 - t_2)$$

$$E[ \cos 2\pi(t_1 + t_2 + 2\phi) ] = 0$$

Option (d)

17. Let $X$ be a random variable which is uniformly chosen from the set of positive odd numbers less than 100. The expectation, $E[X]$ is

[**GATE 2014: 1 Mark**]

**Soln.**

$$E[X] = \frac{1 + 3 + 5 + \ldots - (2n - 1)}{50}$$

Where $n = 50$

$$= \frac{n^2}{50} = 50$$

18. The input to a 1-bit quantizer is a random variable $X$ with pdf $f_X(x) = 2e^{-2x}$ for $x \geq 0$ and $f_X(x) = 0$ for $x < 0$. For outputs to be of equal probability, the quantizer threshold should be_____
Soln. The input to a 1-bit quantizer is a random variable $X$ with pdf

$$ f_X(x) = 2e^{-2x} \quad \text{for } x \geq 0 \quad \text{and} \quad f_X(x) = 0 \quad \text{for } x < 0 $$

let $V_{\text{thr}}$ be the quantizer threshold

$$ V_{\text{thr}} = \int_{-\infty}^{V_{\text{thr}}} 2e^{-2x} \, dx = \int_{0}^{V_{\text{thr}}} 2e^{-2x} \, dx $$

$$ V_{\text{thr}} = \int_{0}^{V_{\text{thr}}} 2e^{-2x} \, dx = \int_{V_{\text{thr}}}^{\infty} 2e^{-2x} \, dx \quad f_X(x) = 0 \quad \text{for } x < 0 $$

$$ \frac{2e^{-2x}}{-2} \bigg|_{0}^{V_{\text{thr}}} = \frac{2e^{-2x}}{-2} \bigg|_{V_{\text{thr}}}^{\infty} $$

$$ (-e^{-2V_{\text{thr}}} + e^{0}) = -(0 - e^{-2V_{\text{thr}}}) $$

$$ e^{-2V_{\text{thr}}} = \frac{1}{2} $$

$$ -2V_{\text{thr}} = \ln \left( \frac{1}{2} \right) = (-0.693) $$

$$ V_{\text{thr}} = \frac{0.693}{2} $$

$$ = 0.346 $$