Z – Transform (Part - 1)

1. The z – transform of the time function

$$\sum_{k=0}^{\infty} \delta(n-k) \ is$$

(a) $(z-1)/z$
(b) $z/(z-1)^2$
(c) $z/(z-1)$
(d) $(z-1)^2/z$

[GATE 1998: 1 Mark]

Soln. Time function is given

$$\boldsymbol{x(n) = \sum_{k=0}^{\infty} \delta\,(n-k)}$$

$$\boldsymbol{= \delta(n) + \delta(n-1) + \delta(n-2) + ---}$$

$$\boldsymbol{x(n) = u(n)}$$

$$\boldsymbol{x(z) = \mathcal{Z}[u(n)] = \frac{z}{(z-1)}}$$

Option (c)

2. The z – transform F(z) of the function $f(nT) = a^{nT}$ is

(a) $\frac{z}{z-a^T}$
(b) $\frac{z}{z+a^T}$
(c) $\frac{z}{z-a^{-T}}$
(d) $\frac{z}{z+a^{-T}}$

[GATE 1999: 1 Mark]

Soln. The z – transform is given by

$$\boldsymbol{\mathcal{Z}[f(nt)] = \sum_{n=-\infty}^{\infty} f(nT)\, z^{-n}}$$

$$= \sum_{n=-\infty}^{\infty} a^{nT} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} (a^T z^{-1})^n$$

$$f(z) = \frac{1}{1 - a^T z^{-1}} = \frac{z}{z - a^T}$$

Option (a)

3. The region of convergence of the z – transform of a unit step function is
   (a) $|z| > 1$
   (b) $|z| < 1$
   (c) (Real part of z) > 0
   (d) (Real part of z) < 0

[GATE 2001: 1 Mark]

Soln. **Given**

$$x(n) = u(n)$$

$$H(z) = \sum_{n=0}^{\infty} u(n)\, z^{-n} = \sum_{n=0}^{\infty} 1.\, z^{-n}$$

$$= \sum_{n=0}^{\infty} (z^{-1})^n$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + - - -$$

$$= \frac{\mathbf{1}}{\mathbf{1 - 1/z}} \quad \textbf{\textit{where}} \quad \frac{\mathbf{1}}{|\mathbf{z}|} < \mathbf{1}$$

So, ROC is the range of value of z for which $|z| > 1$

Option (a)

4. A sequence $x(n)$ with the z – transform $X(z) = z^4 + z^2 - 2z + 2 - 3z^{-4}$ is applied as an input to a linear time – variant system with the impulse response $h(n) = 2\delta(n-3)$ where

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & otherwise \end{cases}$$

The output at n = 4 is

(a) – 6

(b) zero

(c) 2

(d) – 4

[GATE 2003: 1 Mark]

Soln. Given

$$X(z) = z^4 + z^2 - 2z + 2 - 3z^{-4}$$

$$h(n) = 2\delta(n-3)$$

$$h(z) = 2\sum_{n=0}^{\infty} \delta(n-3)\, z^{-n}$$

$$= 2.z^{-3}$$

$$\textit{Note}, \qquad \mathcal{Z}[\delta(n-k)] = z^{-k}$$

Also,

$$y(z) = H(z)\,.X(z)$$

$$= 2z^{-3}(z^4 + z^2 - 2z + 2 - 3z^{-4})$$

$$= 2(z + z^{-1} - 2z^{-2} + 2z^{-3} - 3z^{-7})$$

Taking inverse z transform

$$y(n) = 2[\delta(n+1) + \delta(n-1) - 2\delta(n-2) + 2\delta(n-3) - 3\delta(n-7)]$$

$for\ n = 4$

$$y(4) = 2[\delta(5) + \delta(3) - 2\delta(2) + 2\delta(1) - 3\delta(-3)]$$

At $n = 4, \quad y(4) = 0$

Option (b)

5. The z – transform of a system is

$$H(z) = \frac{z}{z - 0.2}$$

If the ROC is $|z| < 0.2$, then the impulse response of the system is

(a) $(0.2)^n\ u[n]$
(b) $(0.2)^2\ u[-n-1]$
(c) $-(0.2)^2\ u[n]$
(d) $-(0.2)^n\ u[-n-1]$

[GATE 2004: 1 Mark]

Soln. Given

$$H(z) = \frac{z}{(z-0.2)}$$

$$= \frac{z}{z(1-0.2z^{-1})}$$

$$= \frac{1}{(1-0.2z^{-1})}$$

Given ROC is $|z| < 0.2$

Comparing with

$$-a^n u(-n-1) \overset{z}{\longleftrightarrow} \frac{1}{(1 - az^{-1})}$$

So, $h(n) = -(0.2)^n u(-n-1)$ **Option (d)**

6. The region of convergence of z – transform of the sequence $\left(\frac{5}{6}\right)^n u(n) - \left(\frac{5}{6}\right)^n u(-n-1)\ must\ be$

(a) $|z| < \frac{5}{6}$

(b) $|z| > \frac{5}{6}$

(c) $\frac{5}{6} < |z| < \frac{5}{6}$

(d) $\frac{5}{6} < |z| < \infty$

[GATE 2005: 1 Mark]

Soln. For the given sequence we have to find ROC of z – transform.

Given sequence is

$$\boldsymbol{x(n) = \left(\frac{5}{6}\right)^n u(n) - \left(\frac{6}{5}\right)^n u(-n-1)}$$

$$\boldsymbol{first\ term\ of\ the\ sequence\ is}$$

$$\boldsymbol{x(n) = \left(\frac{5}{6}\right)^n u(n)\ \ which\ is\ right\ handed}$$

Sequence and is of the form $\boldsymbol{a^n u(n)}$

Its ROC extends outward.

$$\boldsymbol{ROC:\ |z| > \frac{5}{6}}$$

Second term is the left handed sequences

$\boldsymbol{-\left(\frac{6}{5}\right)^n u(-n-1)\ and\ is\ for\ the\ form - a^n u(-n-1)\ with\ ROC : < \frac{6}{5}}$

So, the combined ROC would be

$$\boldsymbol{\frac{5}{6} < |z| < \frac{6}{5}}$$

Option (c)

7. If the region of convergence of $x_1[n] + x_2[n]$ is $\frac{1}{3} < |z| < \frac{2}{3}$, then the region of convergences of $x_1[n] - x_2[n]$ includes

(a) $\frac{1}{3} < |z| < 3$

(b) $\frac{2}{3} < |z| < 3$

(c) $\frac{3}{2} < |z| < 3$

(d) $\frac{1}{3} < |z| < \frac{2}{3}$

[GATE 2006: 1 Mark]

Soln. Given

ROC of the given sequence

$$\boldsymbol{x_1[n] + x_2[n]}$$

$$\boldsymbol{is\ \ \frac{1}{3} < |z| < \frac{2}{3}\ \ then\ find\ \ ROC\ of\ \ x_1[n] - x_2[n]}$$

The ROC of addition or subtraction of two functions

$\boldsymbol{x_1(n)\ and\ x_2(n)\ \ is\ \ R_1 \cap R_2\ \ .\ \ \ So\ same\ as\ above}$

Option (d)

8. The ROC of z – transform of the discrete time sequence

$$x(n) = \left(\frac{1}{3}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(-n-1)\ is$$

(a) $\frac{1}{3} < |z| < \frac{1}{2}$

(b) $|z| < \frac{1}{2}$

(c) $|z| < \frac{1}{3}$

(d) $2 < |z| < 3$

[GATE 2009: 1 Mark]

Soln. Given

$$\boldsymbol{x(n) = \left(\frac{1}{3}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(-n-1)}$$

Taking z – transform

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n z^{-n} - \sum_{n=-\infty}^{n=-1} \left(\frac{1}{2}\right)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{3} z^{-1}\right)^n - \sum_{n=-\infty}^{-1} \left(\frac{1}{2} z^{-1}\right)^n$$

First term gives ROC $\frac{1}{3} z^{-1} < 1 \quad or \quad |z| > \frac{1}{3}$

Second term gives ROC $\frac{1}{2} z^{-1} > 1 \quad or \quad |z| < \frac{1}{2}$

Thus the combined ROC is common ROC of both terms

$\frac{1}{3} < |z| < \frac{1}{2}$

Option (a)

9. Consider the z – transform $X(z) = 5z^2 + 4z^{-1} + 3; 0 < |z| < \infty$. The inverse z – transform $x[n]$ is
(a) $5\delta[n+2] + 3\delta[n] + 4\delta[n-1]$
(b) $5\delta[n-2] + 3\delta[n] + 4\delta[n+1]$
(c) $5u[n+2] + 3\,u[n] + 4\,u[n-1]$
(d) $5u[n-2] + 3\,u[n] + 4\,u[n+1]$

[GATE 2010: 1 Mark]

Soln. Given

Z – transform $X(z) = 5z^2 + 4z^{-1} + 3$

ROC : $0 < |z| < \infty$

To find inverse z – transform $x[n]$

We know,

$$\boldsymbol{\delta(n \pm a)} \xleftrightarrow{Z} \boldsymbol{z^{\pm a}}$$

$$\boldsymbol{\mathcal{Z}^{-1}[X(z)] = 5\delta[n+2] + 4\delta[n-1] + 3\delta[n]}$$

Option (a)

10. Two discrete time systems with impulse responses $h_1[n] = \delta[n-1]$ and $h_2[n] = \delta[n-2]$ are connected in cascade. The overall impulse response of the cascaded system is

(a) $\delta[n-1] + \delta[n-2]$

(b) $\delta[n-4]$

(c) $\delta[n-3]$

(d) $\delta[n-1]\,\delta[n-2]$

[GATE 2010: 1 Mark]

Soln. Given

$$h_1(n)\delta[n-1] \xleftrightarrow{Z} H_1(z) = z^{-1}$$

$$h_2(n)\delta[n-2] \xleftrightarrow{Z} H_2(z) = z^{-2}$$

Response of the cascaded system is

$$\boldsymbol{H(z) = H_1(z)\,.\,H_2(z)}$$

$$\boldsymbol{z^{-1}\,.\;z^{-2} = z^{-3}}$$

So, $\boldsymbol{h[n] = \delta[n-3]}$

Option (c)

11. If $x[n] = (1/3)^{|n|} - (1/2)^n\, u[n]$, then the region of convergence (ROC) of its Z – transform in the Z – plane will be

(a) $\frac{1}{3} < |z| < 3$

(b) $\frac{1}{3} < |z| < \frac{1}{2}$

(c) $\frac{1}{2} < |z| < 3$

(d) $\frac{1}{3} < |z|$

[GATE 2012: 1 Mark]

Soln. Given

$$\boldsymbol{x[n] = \left(\frac{1}{3}\right)^{|n|} - \left(\frac{1}{2}\right)^{n} u(n)}$$

$$\boldsymbol{= \left(\frac{1}{3}\right)^{n} u(n) + \left(\frac{1}{3}\right)^{-n} u[-n-1] - \left(\frac{1}{2}\right)^{n} u[n]}$$

First term

$$\left(\frac{1}{3}\right)^{n} u(n) \longleftrightarrow \frac{1}{\left(1 - \frac{1}{3} z^{-1}\right)} \; : \; ROC\ |z| > \frac{1}{3}$$

Second term

$$\left(\frac{1}{3}\right)^{-n} u(-n-1) \longleftrightarrow \frac{1}{\left(1 - \frac{1}{3} z^{-1}\right)} \; : \; ROC\ |z| < 3$$

Third term

$$\left(\frac{1}{2}\right)^{n} u(n) \longleftrightarrow \frac{1}{\left(1 - \frac{1}{2} z^{-1}\right)} \; ; \; ROC\ |z| < \frac{1}{2}$$

Overall ROC will be interaction of these ROCs

$\boldsymbol{\frac{1}{2} < |z| < 3}$

Option (c)

12. Let $x[n] = x[-n]$. Let X(z) be the z – transform of $x[n]$. If $0.5 + j0.25$ is a zero of X(z), which one of the following must also be zero of X(z).

(a) $0.5 - j0.25$

(b) $\frac{1}{(0.5+j0.25)}$

(c) $\frac{1}{(0.5-j0.25)}$

(d) $2 + j4$

[GATE 2014: 1 Mark]

Soln. Given $\boldsymbol{x[n] = x[-n]}$

We know

$$x[n] \xleftrightarrow{Z} X[z]$$

$$x[-n] \xleftrightarrow{Z} X[z^{-1}]$$

Time reversal property in z – transform

So, if one zero is (0.5 + j 0.25). Then the other zero will

$$\boldsymbol{be} \quad \frac{\mathbf{1}}{(\mathbf{0.5 + \; j\,0.25})}$$

Option (b)