DTFT, DFT and FFT

1. Let $x(n) = \left(\frac{1}{2}\right)^n u(n), y(n) = x^2(n)\ and\ Y\left(e^{j\omega}\right)$ be the Fourier Transform of $y(n)$. Then $y(e^{j0})$ is

   (a) $\frac{1}{4}$
   (b) 2
   (c) 4
   (d) $\frac{4}{3}$

[GATE 2005: 1 Mark]

Soln. Given

$$x(n) = \left(\frac{1}{2}\right)^n u(n)$$

And $$y(n) = x^2(n) = \left(\frac{1}{2}\right)^{2n} u^2(n)$$

Or, $$y(n) = \left[\left(\frac{1}{2}\right)^2\right]^n u(n) = \left(\frac{1}{4}\right)^n u(n)$$

Taking Fourier Transform

$$Y\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} y(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{4}\right)^n e^{-j\omega n}$$

$$So,\quad Y\left(e^{j0}\right) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{4}\right)^n = 1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + - - - -$$

$$= \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$ **Option (d)**

2. A signal $x(n) = \sin(\omega_0 n + f)$ is the input to a linear time-invariant system having a frequency response $H(e^{j\omega})$. If the output of the system $Ax(n - n_0)$, then the most general form of $\angle H(e^{j\omega})$ will be
(a) $-n_0\omega_0 + \beta$ for any arbitrary real β
(b) $-n_0\omega_0 + 2\pi k$ for any arbitrary integer k
(c) $n_0\omega_0 + 2\pi k$ for any arbitrary integer k
(d) $-n_0\omega_0\phi$

[GATE 2005 : 2 Marks]

Soln. Given

$$\boldsymbol{y(n) = A\ x(n - n_0)}$$

Taking Fourier Transform

$$\boldsymbol{Y(e^{j\omega}) = A\, e^{-j\omega_0 n_0}\ X(e^{j\omega})}$$

Or, $\boldsymbol{H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = A\, e^{-j\omega_0 n_0}}$

Thus $\angle \boldsymbol{H(e^{j\omega}) = -\omega_0 n_0}$

For LTI discrete time system phase and frequency of $\boldsymbol{H(e^{j\omega})}$ are periodic with period $\boldsymbol{2\pi}$. So in general form

$$\boldsymbol{\theta(\omega) = -n_0\omega_0 + 2\pi k}$$

Option (b)

3. A 5-point sequence $x[n]$ is given as
$x[-3] = 1,\ x[-2] = 1,\ x[-1] = 0,\ x[0] = 5, x[1] = 1$
Let $X(e^{j\omega})$ denote the discrete-time Fourier Transform of $x[n]$. The value of

$$\int_{-\pi}^{\pi} X(e^{j\omega})\, d\omega \quad is$$

(a) 5

(b) 10 π

(c) 16 π

(d) $5 + j\ 10\pi$

[GATE 2007 : 2 Marks]

Soln. Discrete Fourier Transform (DTFT) when $N \to \infty$ is given by

$$\boldsymbol{x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j\omega}\right) e^{j\omega n}\, d\omega}$$

For n = 0, we get

$$\boldsymbol{x[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j\omega}\right) e^{j\omega.0}\, d\omega}$$

$$\boldsymbol{= \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j\omega}\right) d\omega}$$

$$\boldsymbol{or, \quad \int_{-\pi}^{\pi} X\left(e^{j\omega}\right) d\omega = 2\pi\, x[0] = 2\pi \times 5}$$

Option (b)

4. $\{x(n)\}$ is real-valued periodic sequence with a period N. $x(n)$ and $X(k)$ form N-point Discrete Fourier Transform (DFT) pairs. The DFT Y(k) of the sequence

$$y(n) = \frac{1}{N} \sum_{r=0}^{N-1} x(r)\ X(n + r)\ is$$

(a) $|X(k)|^2$

$$(b)\ \frac{1}{N}\sum_{r=0}^{N-1} X(r)X^*\ (k+r)$$

$$(c)\ \frac{1}{N}\sum_{r=0}^{N-1} X(r)X\ (k+r)$$

(d) 0

[GATE 2008 : 2 Marks]

Soln. Given

$$\boldsymbol{y(n) = \frac{1}{N}\sum_{r=0}^{N-1} x(r)\ x(n+r)}$$

$\boldsymbol{y(n)}$ is the correlation of a signal $\boldsymbol{x(n)}$ with itself.

The Fourier Transform of auto correlation function is $\boldsymbol{|X(k)|^2}$

Option (b)

5. The 4-point discrete Fourier Transform (DFT) of a discrete time sequence {1, 0, 2, 3} is

(a) $[0, -2+2j, 2, -2-2j]$

(b) $[2,\ 2+2j,\ 6,\ 2-2j]$

(c) $[6,\ 1-3j, 2,\ 1+3j]$

(d) $[6, -1+3j,\ 0, -1-3j]$

[GATE 2009 : 2 Marks]

Soln. Given discrete time sequence

$\boldsymbol{x[n] = \{1, 0, 2, 3\}\ and\ N = 4}$

$$X[k] = \sum_{n=0}^{N-1} x[n]\ e^{-j2\pi nk/N} \qquad k = 0, 1, --(N-1)$$

For N = 4

$$X[k] = \sum_{n=0}^{3} x[n]\ e^{-j2\pi nk/4} \qquad k = 0, 1, --3$$

Now

$$X[k] = \sum_{n=0}^{3} x[n] = x[0] + x[1] + x[2] + x[3]$$

$$= 1 + 0 + 2 + 3 = 6$$

$$x[1] = \sum_{n=0}^{3} x[n]\, e^{-\frac{j\pi n}{2}} = x[0] + x[1]e^{-\frac{j\pi}{2}} + x[2]e^{-j\pi} + x[3]e^{-j\pi 3/2}$$

$$= 1 + 0 - 2 + j3 = -1 + j3$$

$$x[2] = \sum_{n=0}^{3} x[n]\, e^{-j\pi n} = x[0] + x[1]e^{-j\pi} + x[2]e^{-2j\pi} + x[3]e^{-j3\pi}$$

$$= 1 + 0 + 2 - 3 = 0$$

$$x[3] = \sum_{n=0}^{3} x[n]\, e^{-j3\pi n/2} = x[0] + x[1]e^{-j3\pi/2} + x[2]e^{-j3\pi} + x[3]e^{-j3\pi/2}$$

$$= 1 + 0 - 2 - j3 = -1 - j3$$

Thus $[6, -1 + j3, 0, -1 - j3]$ **Option (d)**

6. For an N-point FFT algorithm with $N = 2^m$, which one of the following statements is TRUE?
   (a) It is not possible to construct is signal flow graph with both input and output in normal order
   (b) The number of butterflies in the m^{th} state is N/m
   (c) In-place computation requires storage of only 2N node data
   (d) Computation of a butterfly requires only one complex multiplication

[GATE 2010 : 1 Mark]

Soln. For an N-point FFT algorithm.

Butterfly operate on one pair of samples and involves two complex additions and one complex multiplication

Option (d)

7. The first six points of the 8-point DFT of a real valued sequence are $5, 1 - j3, 0, 3 - 4j$, and $3 + j4$. The last two points of the DFT are respectively
   (a) $0, 1 - j3$
   (b) $0, 1 + j3$
   (c) $1 + j3, 5$
   (d) $1 - j3, 5$

[GATE 2011 : 2 Marks]

Soln. Given that the sequence is real valued with 8 points.

i.e.

$$\boldsymbol{X(k) = [X(0), X(1), X(2), X(3), X(4), X(5), X(6), X(7)]}$$

$$\boldsymbol{= [5, (1 - j3), 0, (3 - j4), 0, (3 + j4), -, -]}$$

If the sequence $x[n]$ is real then $X[k]$ is conjugate symmetric

i.e. $\quad \boldsymbol{X[k] = X^*[N - k]}$

thus $\quad \boldsymbol{X[6] = X^*[8 - 6] = X^*[2] = 0}$

$$\boldsymbol{X[7] = X^*[8 - 7] = X^*[1] = 1 + j3}$$

Option (b)

8. Consider a discrete time periodic signal $x[n] = \sin\left(\frac{\pi n}{5}\right)$. Let a_k be the complex. Fourier series coefficients of $x[n]$. The coefficients $\{a_k\}$ are non-zero when $k = Bm \pm 1$, where m is any integer. The value of B is ______.

[GATE 2014 : 2 Marks]

Soln. Given

$$x(n) = \sin\left[\frac{\pi n}{5}\right]$$

Find the value of B

$$x(n) = \sin\left[\frac{\pi n}{5}\right]$$

Time period of $x(n) = \frac{2\pi}{\omega_0} = \frac{2\pi}{\pi/5} = 10$

$$\sin\left[\frac{\pi}{5}n\right] = \frac{1}{2j}.e^{j\frac{\pi}{5}n} - \frac{1}{2j}e^{-j\frac{\pi}{5}n}$$

So, α_k **exist for** $K = \pm 1$

$x[n]$ **is a discrete time signal its Fourier series coefficients exist after each time interval value**

$i.e.$ $K = \pm 1, 10 \pm 1, 20 \pm 1, ----$

$$\alpha_k = 10m \pm 1 = Bm \pm 1$$

$i.e$ $B = 10$

Answer: B = 10