Z – Transform (Part - II)

1. The Z – Transform of the following real exponential sequence
\[ x(nT) = a^n, nT \geq 0 \]
\[ = 0, nT < 0, \ a > 0 \]
(a) \( \frac{1}{1-z^{-1}}; \ |z| > 1 \)
(b) \( \frac{1}{1-az^{-1}}; \ |z| > a \)
(c) 1 for all \( z \)
(d) \( \frac{1}{1-az^{-1}}; \ |z| < a \)

[GATE 1990: 2 Marks]

Soln. The given sequence can be written as
\[ x(n)a^n(n) \text{ where } a > 0 \]
Z – Transform of given sequence is
\[ X(z) = \frac{z}{z-a}; \ |z| > |a| = \frac{1}{1-az^{-1}}, \ |z| > a \]
Option (b)

2. A linear discrete-time system has the characteristics equation,
\[ z^3 - 0.81z = 0 \]. The system
(a) is stable
(b) is marginally stable
(c) is unstable
(d) stability cannot be assessed from the given information

[GATE 1992: Marks]

Soln. Characteristic equation is given
\[ z^3 - 0.81z = 0 \]
\[ z(z^2 - 0.81) = 0 \]
\[ z(z - 0.9)(z + 0.9) = 0 \]
So, poles are
\[ z = 0, \ 0.9 \ \text{and} \ -0.9 \]
Note that all three poles are inside the unit circle, so the system is stable
Option (a)
3. The z – transform of a signal is given by
\[ C(z) = \frac{1}{4} \frac{z^{-1}(1 - z^{-4})}{(1 - z^{-1})^2} \] 
Its final value is

(a) 1/4  \hspace{1cm} (c) 1.0
(b) zero \hspace{1cm} (d) infinity

\[ \text{[GATE 1999: 2 Marks]} \]

Soln. Final value theorem for Z – Transform is

\[ \lim_{N \to \infty} x[N] = \lim_{z \to 1} (1 - z^{-1}) X(z) \]
\[ = \lim_{z \to 1} \frac{1}{4} \frac{(1-z^{-1}) z^{-1} (1-z^{-4})}{(1-z^{-1})^2} \]
\[ = \lim_{z \to 1} \frac{1}{4} \frac{z^{-1}(1-z^{-4})}{(1-z^{-1})} \]
\[ = \lim_{z \to 1} \frac{1}{4} \frac{1}{z^{4}} \frac{(z^4-1)/z^4}{(z-1)/z} \]
\[ = \lim_{z \to 1} \frac{1}{4} \frac{(z^2-1)(z^2+1)}{z^4(z-1)} \]
\[ = \lim_{z \to 1} \frac{1}{4} z^2 \frac{(z+1)(z-1)(z^2+1)}{(z-1)} \]
\[ = \lim_{z \to 1} \frac{1}{4} z^2 (z + 1)(z^2 + 1) = \frac{1}{4} \cdot 1^2 \cdot 2 \cdot 2 = 1 \]

Option (a)

4. If the impulse response of a discrete-time system is
\[ h[n] = -5^n u[-n - 1] \] Then the system function H(z) is equal to

(a) \[ \frac{-z}{z-5} \] and the system is stable
(b) \[ \frac{z}{z-5} \] and the system is stable
(c) \[ \frac{-z}{z-5} \] and the system is unstable
(d) \[ \frac{z}{z-5} \] and the system is unstable

\[ \text{[GATE 2002: 2 Marks]} \]
Soln. Impulse response \( h(n) = -5^n u[-n - 1] \)

Above response is having left handed sequence whose \( z \)-transform has standard form

\[
-a^n u(n)(-n - 1) \quad \overset{z}{\longleftrightarrow} \quad \frac{1}{1 - a z^{-1}}; \quad ROC : |z| < a
\]

Thus

\[
-5^n u(n)(-n - 1) \quad \overset{z}{\longleftrightarrow} \quad \frac{1}{1 - 5 z^{-1}}; \quad ROC : |z| < 5
\]

or,

\[
-5^n u(n)(-n - 1) \quad \overset{z}{\longleftrightarrow} \quad \frac{z}{z - 5}; \quad ROC : |z| < 5
\]

since ROC is \(|z| < 5\) so it includes unit circle, system is stable

Option (b)

5. A causal LTI system is described by the difference equation

\[
2y[n] = \alpha y[n - 2] - 2x[n] + \beta x[n - 1]
\]

The system is stable only if

(a) \(|\alpha| = 2, \ |\beta| < 2\)
(b) \(|\alpha| > 2, \ |\beta| > 2\)
(c) \(|\alpha| < 2, \ any \ value \ of \ \beta\)
(d) \(|\beta| < 2, \ any \ value \ of \ \alpha\)

[GATE 2004: 2 Marks]

Soln. Casual LTI system is described by the difference equation

\[
2 \ y[n] = \alpha \ y[n - 2] - 2 \ x[n] + \beta \ x[n - 1]
\]

Taking \( z \)-transform

\[
2 \ Y[z] = \alpha \ Y(z) z^{-2} - 2 \ X(z) + \beta \ X(z) z^{-1}
\]

\[
\frac{Y(z)}{X(z)} = \frac{(\beta \ z^{-1} - 2)}{(2 - \alpha \ z^{-2})}
\]
Or, \[ H(z) = \frac{z(\beta/2-z)}{z^2-\alpha/2} \]

It has poles at \( \pm \sqrt{\alpha/2} \) and zero at 0 and \( \beta/2 \)

For stable system poles must lie inside the unit circle of \( z \)-plane.
But 0 can lie anywhere in plane. Thus \( \beta \) can be of any value

Option (c)

6. The \( z \)-transform \( X[z] \) of a sequence \( x[n] \) is given by \( X[z] = \frac{0.5}{1-2z^{-1}} \). It is given that the Region of convergence of \( X[z] \) includes the unit circle. The value of \( x[0] \) is

(a) -0.5  (c) 0.25
(b) 0  (d) 0.5

[\text{GATE 2007: 2 Marks}]

\text{Soln. Given}

\[ X(z) \text{ of a sequence } x[n] \]

\[ X(z) = \frac{0.5}{1-2z^{-1}} \]

Above transform is for left handed sequence with

\( \text{ROC} : |z| < 2 \)

Corresponding sequence is

\[ x(n) = -(0.5)2^{-n}u(-n-1) \]

So, \( x(0) = 0 \)

If, the value of \( x(0) \) is determined by initial value theorem then it will not be correct, since the sequence \( x(n) \) is defined for \( n < 0 \)

Option (b)

7. A system with transfer function \( H(z) \) has impulse response \( h(n) \) defined as \( h(2) = 1, h(3) = -1 \) and \( h(k) = 0 \) otherwise. Consider the following statements.

\( S_1 \) : \( H(z) \) is low-pass filter
S2 : H(z) is an FIR filter
Which of the following is correct?
(a) Only S2 is true
(b) Both S1 is and S2 are false
(c) Both S1 is and S2 are true, and S2 is a reason for S1
(d) Both S1 is and S2 are true, and S2 is not a reason for S1

[GATE 2009: 2 Marks]

Soln. Given
\[ h(2) = 1 \]
\[ h(3) = -1 \]
\[ h(k) = 0 \text{ otherwise} \]

The diagram of the response is as shown

Note, that it has the finite magnitude values. So it is having finite impulse response.

So, FIR

But it is not low pass filter So, S1 is false

Option (a)

8. The transfer function of a discrete time LTI system is given by

\[ H(z) = \frac{2 - \frac{3}{4}z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} \]

Consider the following statements:

S1 : The system is stable and casual tor

\[ ROC : |z| > \frac{1}{2} \]
The system is stable but not causal for

\[ ROC : |z| < \frac{1}{4} \]

The system is neither stable nor causal

For \( ROC : \frac{1}{4} < |z| < \frac{1}{2} \)

Which one of the following statements is valid?

(a) Both \( S_1 \) and \( S_2 \) are true (c) Both \( S_1 \) and \( S_3 \) are true
(b) Both \( S_2 \) and \( S_3 \) true (d) \( S_1 \), \( S_2 \) and \( S_3 \) are all true

[GATE 2010: 2 Marks]

Soln.

We know:

(i) Causal System : A discrete time LTI system is causal if and only if the ROC of its system function is exterior of a circle, including infinity

(ii) Stable System : A discrete time LTI system is stable if and only if the ROC of its system function includes the unit circle i.e. \( |z| = 1 \)

\[
H(z) = \frac{2 - \frac{3}{4} z^{-1}}{1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}}
\]

\[
= \frac{(1 - \frac{1}{4} z^{-1}) + (1 - \frac{1}{2} z^{-1})}{(1 - \frac{1}{4} z^{-1})(1 - \frac{1}{2} z^{-1})}
\]

\[
H(z) = \frac{1}{(1 - \frac{1}{4} z^{-1})} + \frac{1}{(1 - \frac{1}{2} z^{-1})}
\]

\[ ROC : |z| > \frac{1}{2} \]

\[
h(n) = \left(\frac{1}{4}\right)^n u(n) + \left(\frac{1}{2}\right)^n u(n)
\]

The system is causal, since ROC is exterior of the circle.

ROC of \( H(z) \) includes unit circle so it is stable

So, \( S_1 \) is true

For ROC  : \( |z| < \frac{1}{4} \)
ROC does not include the unit circle. So system is not stable.

So, $S_2$ is not true

For ROC : \( \frac{1}{4} < |z| < \frac{1}{2} \)

ROC does not include unit circle.

So system is not stable. Also ROC is not exterior of \(|z| = \frac{1}{2}\). So it not causal

So, $S_3$ is true

Both $S_1$ and $S_3$ are true

9. Two systems $H_1(z)$ and $H_2(z)$ are connected in cascade as shown below. The overall output $y(n)$ is the same as the input $x(n)$ with a one unit delay. The transfer function of the second system $H_2(z)$ is

![Cascade System Diagram]

\[
x(n) \rightarrow H_1(z) = \frac{(1 - 0.4z^1)}{(1 - 0.6z^{-1})} \rightarrow H_2(z) \rightarrow y(n)
\]

(a) \( \frac{(1-0.6z^{-1})}{z^{-1}(1-0.4z^{-1})} \)

(b) \( \frac{z^{-1}(1-0.6z^{-1})}{(1-0.4z^{-1})} \)

(c) \( \frac{z^{-1}(1-0.4z^{-1})}{(1-0.6z^{-1})} \)

(d) \( \frac{(1-0.4z^{-1})}{z^{-1}(1-0.6z^{-1})} \)

\[ [GATE 2011: 2 Marks] \]

Soln. Given

\[ y[n] = x[n - 1] \]

Taking Z Transfer on both sides

\[ Y(z) = z^{-1} X(z) \]

or \( \frac{Y(z)}{X(z)} = z^{-1} \)

For cascaded system

\[ H(z) = H_1(z).H_2(z) \]
\[ z^{-1} = \frac{(1-0.4 z^{-1})}{(1-0.6 z^{-1})} \cdot H_z(z) \]

Or, \[ H_2(z) = z^{-1} \frac{(1-0.6 z^{-1})}{(1-0.4 z^{-1})} \]

Option (b)

10. Let \[ x[n] = \left(-\frac{1}{9}\right)^n u(n) - \left(-\frac{1}{3}\right)^n u(-n-1) \]. The Region of Convergence (ROC) of the Z – Transform of \( x[n] \)

(a) \[ \text{is } |z| > \frac{1}{9} \]
(b) \[ \text{is } |z| < \frac{1}{3} \]
(c) \[ \text{is } \frac{1}{3} > |z| > \frac{1}{9} \]
(d) \[ \text{does not exist} \]

[GATE 2014: 2 Marks]

Soln. Given

\[ x[n] = \left(-\frac{1}{9}\right)^n u(n) - \left(-\frac{1}{3}\right)^n u(-n-1) \]

Let, \[ x_1(n) = \left(-\frac{1}{9}\right)^n u(n) \] and
\[ x_2(n) = \left(-\frac{1}{3}\right)^n u(-n-1) \]
\[ x_1(n) = \left(-\frac{1}{9}\right)^n u(n) \]

It is right sided sequence.

\[ X_1(z) = \frac{1}{1 - \left(-\frac{1}{9}\right)z^{-1}} \]

ROC : \[ |z| \geq \frac{1}{9} \text{ or } |z| > \frac{1}{9} \]
\[ x_2(n) = \left(-\frac{1}{3}\right)^n u(-n-3) \]

It is left sided sequence

\[ X_2(z) = \frac{1}{1 - \left(-\frac{1}{3}\right)z^{-1}} \]

ROC : \[ |z| \leq \frac{1}{3} \]
When both these transform are added then

ROC must be between

\[
\frac{1}{3} > |z| > \frac{1}{9}
\]

Option (c)

11. The input-output relationship of a causal stable LTI system is given as

\[y[n] = \alpha y[n - 1] + \beta x[n]\]

If the impulse response \(h[n]\) of this system satisfies the condition

\[\sum_{n=0}^{\infty} h[n] = 2,\]

the relationship between \(\alpha\) and \(\beta\) is

(a) \(\alpha = 1 - \beta/2\)
(b) \(\alpha = 1 + \beta/2\)
(c) \(\alpha = 2\beta\)
(d) \(\alpha = -2\beta\)

[GATE 2014: 2 Marks]

Soln. Given input output relation for causal, stable, LTI system.

\[y[n] = \alpha y[n - 1] + \beta x[n]\]

Taking Z – Transfer

\[Y(z) = \alpha z^{-1} Y(z) + \beta X(z)\]

\[Y(z)[1 - \alpha z^{-1}] = \beta X(z)\]

\[\frac{Y(z)}{X(z)} = H(z) = \frac{\beta}{1 - \alpha z^{-1}}\]

\[h(n) = \beta \cdot \alpha^n u[n]\]

Also,

\[\sum_{n=0}^{\infty} h(n) = 2 \quad \text{i.e.} \quad \sum_{n=0}^{\infty} \beta \alpha^n u[n] = 2\]

\[\frac{\beta}{1 - \alpha} = 2 \quad \text{or} \quad \beta = 2 - 2\alpha \quad \text{or} \quad \alpha = 1 - \beta/2\]

Option (a)
11. The $Z$-Transform of the sequence $X[n]$ is given by $X(z) = \frac{1}{(1-2z^{-1})^2}$, with the region of convergence $|z| > 2$. Then, $X[2]$ is _________

Soln. Given

$$X(z) = \frac{1}{(1-2z^{-1})^2} = |z| > 2$$

Or, $X(z) = (1 - 2z^{-1})^{-2}$

$$= 1 + 4 z^{-1} + 12 z^{-2} + \ldots \ldots$$

Expanding by binomial expansion

Taking inverse $z$-transform

We get, $x[n] = \{1, 4, 12, \ldots \}$

So, $x[2] = 12$